

(Quantum) Fractional Brownian Motion and Multifractal Processes under the Loop of a Tensor Networks

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We derive fractional Brownian motion and stochastic processes with multifractal properties using a framework of network of Gaussian conditional probabilities. This leads to the derivation of new representations of fractional Brownian motion. These constructions are inspired from renormalization. The main result of this paper consists of constructing each increment of the process from two-dimensional gaussian noise inside the light-cone of each separate increment. Not only does this allows us to derive fractional Brownian motion, we can introduce extensions with multifractal flavour. In another part of this paper, we discuss the use of the multi-scale entanglement renormalization ansatz (MERA), introduced in the study critical systems in quantum spin lattices, as a method for sampling integrals with respect to such multifractal processes. After proper calibration, a MERA promises the generation of a sample of size N of a multifractal process in the order of $O(N \log(N))$, an improvement over the known methods, such as the Cholesky decomposition and the circulant methods, which scale between $O(N^2)$ and $O(N^3)$.

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I. INTRODUCTION

The study of long memory processes [2] is an old story. When sampling the same population at different point in times, X_1, \dots, X_n , we often hope and assume that the time-average reflects the local average,

$$\bar{X} = \frac{1}{n} \sum_j X_j, \quad E(\bar{X}) \approx \frac{1}{n} E(X_j)$$

The central limit theorem tells us that this is indeed true if X_j are identically distributed and are independent. The variance is then inversely proportional with time, the sample size, and thus decay. This fact remains true even when X_j have weak correlations such as an exponential decay [1, 15]. This universality of the inverse proportionality with the sample size, breaks down when the correlations are stronger and decay polynomially. This phenomena of long range dependence is very well studied in many areas of physics, from statistical physics to quantum field theory.

In statistics, many classes of gaussian processes with long-range dependency have been studied throughout history. One of the most studied and well known is certainly fractional Brownian motion. These processes were first introduced by Kolmogorov in 1940 when investigating turbulences. Perhaps it is Benoit Mandelbrot who truly recognized the importance of the fractional Brownian motion. In 1965, he published his insights on the work of the hydrologist Harold Erwin Hurst, who observed discrepancies in the yearly variation of the levels of the Nile river [8, 9, 11]. Using a fractional integration of the

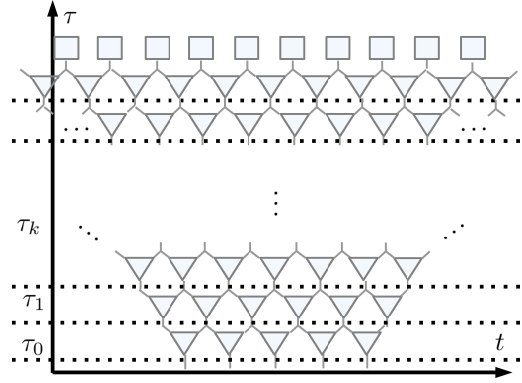


FIG. 1: Network representation of the joint probability distribution of a stochastic process.

Brownian motion, the following process $B_H(t)$ is introduced,

$$B_H(t) = \int_{-\infty}^0 \{(t-s)^{H-1/2} - (-s)^{H-1/2}\} dB(s) + \int_0^t (t-s)^{H-1/2} dB(s) \quad (1)$$

The constant $H \in (0, 1)$ is also known as the Hurst index. For $H = 1/2$ this reduces to regular Brownian motion.

With the rise of computational power, new methods were developed for simulating condensed matter. The challenge faced in such system was the exponentially growing number of parameters. Solutions were presented in the form of various ansatz states [13, 16]. These states were constructed from networks of tensors with a particular geometry.

In 2005, Vidal presented a tensor network ansatz reminiscent of renormalization to simulate quantum critical systems [17]. While being mostly used for numerics, such scheme sparks various interests in other areas such as high energy physics [12] etc... A continuum version was presented in 2010

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by Haegemann et. al. [6].

In this work, we show that fractional brownian motion can be related with such networks (1,6). We start from a discrete process $(X_n)_{n=1}^N$ with a joint probability distribution $p(X_1, \dots, X_N)$ represented by a certain network. Under renormalization of the parameters of the network, we prove that the process $\sum_n X_n$ is precisely a fractional Brownian motion. The networks also yield a new representation of fractional Brownian motion.

In the first part, we construct a new network with an underlying causal structure which follows from a renormalization flow. We will show that this network generates fractional brownian motion. The starting point of this work is the relationship between fractional brownian motion and renormalization [7]. Based on this knowledge, we also discuss the use of MERA for simulating such processes. While a MERA is more challenging to derive analitically, we can calibrate the parameters numerically. It turns out that the MERA structure offers the possibility of simulating such gaussian processes more efficiently, $O(N \log N)$, than the regular methods such as the Choleksy decompositon or circulant methods [3].

The two networks (6,1) presented in this work differ in the direction of the renormalization flow. While for MERA the real space is renormalized in the virtual space, this is quite the opposite in the second network.

II. NETWORK REPRESENTATION OF STOCHASTIC PROCESSES

By compounding or integrating familar processes, Statiicians derive new ones with new desired properties. The use of processes with simple distributions, such as Gaussian distributions, also permits the efficient simulations of the processes without having the derive the distribution of the new process. Sometimes, however, the opposite direction is necessary. This allows for a change of measure, which can further simplify the process. The most famous example is the so-called Girsanov theorem, which allows to eliminate the drift from a Brownian motion by change of measure [5].

Time series [2] such as ARIMA, ARFIMA, etc... , are succesful techniques for tackling memory. These processes make use of the increments of a one-dimensional Brownian motion up to some time t ,

$$X_t = \int_0^t c(s) dB_s$$

At each later time, new increments are added. In the era of tensor networks, besides the real, here time-, axis additional virtual dimensions are added. These new axes potentially and seemingly add new parameters, but present us with new insights in such processes. The goals of this section is to discuss the expansion of the joint probability distribution $P(X_1, \dots, X_N)$ of the increments \vec{X}_j of a process $Y_N = \sum_j X_j$ as a circuit of conditional probabilities. The circuit is set up along a virtual dimension, which we denote as τ . The final output of the circuit is at τ_0 , which is then the

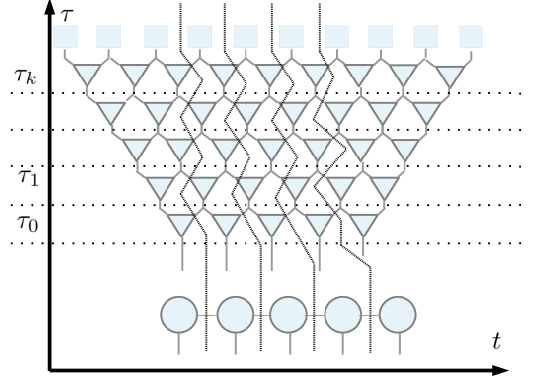


FIG. 2: The joint probability distribution can be approximated by a finite debt circuit. This in turn can be represented by a so-called Matrix Product State (5). Each of the local matrices of Matrix Product States have a covariance matrix of finite dimension.

seeked probability.

$$P(\vec{X}_{t_j, \tau_0}) = \sum_{j_1, \dots, j_{N_\infty}} P(\vec{X}_{t_{j_0}, \tau_0} | \vec{X}_{t_{j_1}, \tau_1}) \\ P(\vec{X}_{t_{j_1}, \tau_1} | \vec{X}_{t_{j_2}, \tau_2}) \dots P(\vec{X}_{t_{j_{N_\infty-1}}, \tau_{N_\infty-1}} | \vec{X}_{t_{j_{N_\infty}}, \tau_{N_\infty}})$$

Both networks are intrinsically connected with renormalization. The first circuit (1) has a natural causal structure and is discussed in the next section. We will also display the power of an already known network, MERA, shown in figure (6). Our approach in section (II B) is then purely numerical.

A. A Light-Cone network for fractional Brownian Motion

In 1966, Leo P. Kadanoff proposed the "block-spin" renormalization group in his study of phase transitions of the Ising model. He hypothesized that because spins would line up in large blocks near the critical points, then neighbouring spins can be regrouped and treated as a single entity. This ansatz allowed him to rederive scaling laws near the critical point.

One of the properties of Fractional Brownian motion is the self-similarity of the process,

$$B^{\text{fm}}(at) \stackrel{d}{=} |a|^H B^{\text{fm}}(t)$$

where the equality is understood as in distribution. This scaling property fits perfectly with the philosophy of Kadanoff's renormalization on a tree. If some parameter $m_{\tau_k}^\tau$ describes some group of tensors from a virtual time τ_k to τ , then according to Kadanoff it should be a related by a rescaling when the block is made larger,

$$m_{\tau_k}^{a\tau} \propto a^{H'} m_{\tau_k}^\tau$$

Keeping these ideas in mind, we elaborate on the following arrangement of tensors pictured in figure (1). The output in the horizontal axis is the joint probability of the increments of the process at each time t . The tensors are contracted along the vertical axis, which we will call the virtual time τ .

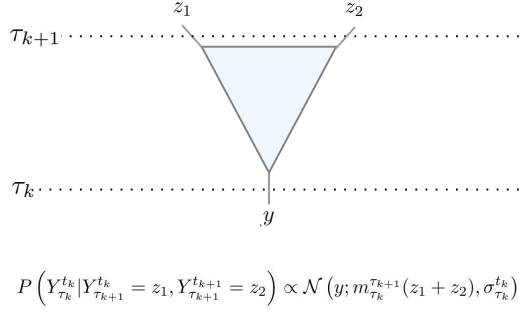


FIG. 3: Tensor representation of the gaussian conditional probabilities used for the construction of (1).

We start from a discretized construction. In order to derive a continuum limit, we need to refine the lattice spacing in the real and virtual time with the sample size N . It turns out the following choice of tree tensors yields fractional Brownian motion,

$$P(Y_{\tau_k}^{t_k} | Y_{\tau_{k+1}}^{t_{k+1}} = z_1, Y_{\tau_{k+1}}^{t_{k+1}} = z_2) \propto \mathcal{N}(y; m_{\tau_k}^{\tau_{k+1}}(z_1 + z_2), \sigma_{\tau_k}^{t_k}) \quad (2)$$

These components are illustrated graphically in figure (3), with the affine parameter $m_{\tau_k}^{\tau_{k+1}}$,

$$m_{\tau_k}^{\tau_{k+1}} = \gamma_k^{k+1} \exp\left([1-H] \int_{\tau_k}^{\tau_{k+1}} ds \frac{1}{s}\right), \quad k > 0$$

The normalization coefficient γ_n^{n+1} is given by,

$$\gamma_n^{n+1} = \frac{\left[(n+1) \binom{2(n+1)}{n}\right]^{-1/2}}{\left[n \binom{2n}{n}\right]^{-1/2}}$$

and the variance σ_{τ_k} ,

$$\sigma_{\tau_k} = \frac{1}{2} \sqrt{H|2H-1|\Gamma(1-H)^{-1}} \sigma$$

We introduce a cutoff on the lower boundary which depends on ϵ ,

$$m_{\tau_0}^{\tau_1} = \frac{1}{2} \exp\left([1-H] \int_{\epsilon^{1/|1-H|}}^{\tau_1} ds \frac{1}{s}\right), \quad k > 0$$

with variance,

$$\sigma_{\tau_0} = \begin{cases} \frac{\sqrt{\epsilon}}{2} \sqrt{H|2H-1|\Gamma(1-H)^{-1}} \sigma & H = 1/2 \\ \frac{1}{2} \sqrt{H|2H-1|\Gamma(1-H)^{-1}} \sigma & \text{else} \end{cases}$$

The time interval $[0, T]$ is divided in N segments of length ϵ . Simultaneously, the virtual dimension is discretized, however, not uniformly,

$$\tau_n = \sqrt{n\epsilon^2}$$

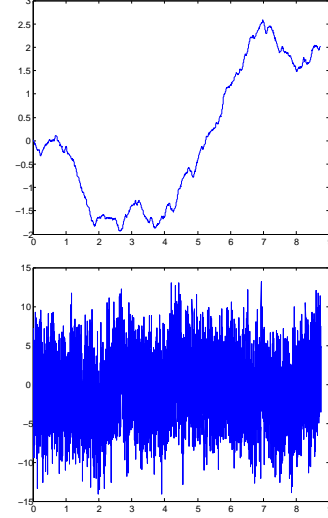


FIG. 4: Plot of the path and increments of a fractional Brownian motion generated using MERA.

In the limit, $N \rightarrow \infty$ and $\epsilon \rightarrow 0$, we show that this joint probability distribution indeed represents fractional Brownian motion. Further details are given in the appendix (A).

Theorem 1. *The process $B_T^{fm} = \sum_j X_{t_j}$ with joint probability distribution $P(X_{t_1}, \dots, X_{t_N})$ constructed earlier, is a fractional brownian motion with Hurst index H in the limit $N \rightarrow \infty$ and $t_N \rightarrow T$.*

Proof. Since the process is gaussian, it is sufficient to show that,

$$E[B_t^{fm} B_s^{fm}] = \frac{1}{2} \sigma^2 (t^{2H} + s^{2H} - |t-s|^{2H}), \quad s < t < T$$

One should see that the following relation for $H \neq 1/2$,

$$\begin{aligned} \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}) \\ = H(2H-1) \int_0^t du_1 \int_0^s du_2 |u_1 - u_2|^{2H-2} \end{aligned}$$

Using this relation and noticing that,

$$E(X_{t_j} X_{t_k}) \approx \sigma \epsilon^2 H(2H-1) |t_j - t_k|^{2H-2}$$

the claim follows. \square

B. MERA: Sampling integrals of fractional Brownian motion

Clearly, the integral representation (1) is very difficult to sample. There exists however a few methods [3] for sampling fractional brownian motion. An even-more challenging problem is the sampling of an integral with respect to fractional

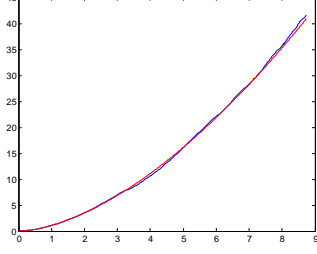


FIG. 5: A fit of the correlation $E(B_t^2)$ for fractional processes generated by a MERA. The correlation is seen to grow as $|t|^{1.66}$.

Brownian motion.

$$\int_0^t f(s) dB^{\text{fm}}(s)$$

Networks seem to present an elegant alternative solution to this problem. As we can easily sample X_{t_j} , so that,

$$\sum_j f(t_j) X_{t_j} \rightarrow \int_0^t f(s) dB_s^{\text{fm}} \quad (3)$$

converges to the desired result for sufficiently large N . The sampling of the sum in equation (3) can be done in the following way. As often used in control theory, the transition tensor $P(Y_{\tau_k} | Z_{\tau_{k+1}} = z)$ in equation (2) can be shown to be equivalent to the algebraic equation,

$$Y_{\tau_k}^{t_k} = m_{\tau_k}^{\tau_{k+1}}[t_k](Z_{\tau_{k+1}}^{t_k} + Z_{\tau_{k+1}}^{t_{k+1}}) + \xi_{\tau_k}^{t_k} \quad (4)$$

using the additional normally distributed random variable $\xi_{\tau_k} \propto \mathcal{N}(0, \sigma_{\tau_k})$

If we combine and trace the vector in equation (4) with variables (c_1, c_2) , we derive a discretized renormalization flow equation,

$$\begin{aligned} \sum_{j=N_1}^{N_2} c_j Y_{\tau_k}^{t_j} &= \sum_{j=N_1}^{N_2} c_j \xi_{\tau_k}^{t_j} + m_{\tau_k}^{\tau_{k+1}} \sum_{j=N_1+1}^{N_2} (c_{j-1} + c_j) Y_{\tau_{k+1}}^{t_j} \\ &+ m_{\tau_k}^{\tau_{k+1}} \left(c_{t_{N_1}} Y_{\tau_{k+1}}^{t_{N_1}} + c_{t_{N_2+1}} Y_{\tau_{k+1}}^{t_{N_2}} \right) \end{aligned}$$

As we go higher up the tree, we replace the sum of the variables on each branch by a sum over the local fluctuations and we renormalize the terms which are connected by a higher branch.

For example, we readily derive that for $H \neq 1/2$,

$$\begin{aligned} B_T^{\text{fm}} &\approx \alpha_H \sigma \sum_{n=0}^{N_\infty} m_{\tau_0}^{\tau_n} \sum_{k=0}^{N+n} \left(\theta \left(a(k \leq \lceil \frac{n}{2} \rceil) \right) \binom{n}{k} \right. \\ &+ \theta \left(k \geq N + \lceil \frac{n}{2} \rceil + 1 \right) \binom{n}{k - (N + \lceil \frac{n}{2} \rceil + 1)} \Bigg) \\ &\quad \theta \left(\lceil \frac{n}{2} \rceil + 1 \leq k \leq N + \lceil \frac{n}{2} \rceil \right) \binom{n}{k} \Bigg) \xi_{\tau_k}^{\tau_n} \end{aligned}$$

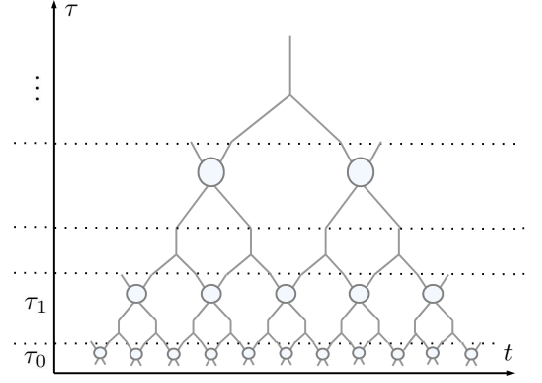


FIG. 6: The multi-scale entanglement renormalization ansatz. The end of the circuit represents a quantum states, and in our paper the joint probability distribution of a process.

We denoted N_∞ as the debt of the circuit. Naturally as proven this should be as large as possible, $N_\infty \rightarrow \infty$. It turns out that such finite debt circuit are related to so-called Matrix Product States which we discuss in the next section. Unfortunately, the circuit (1) presented earlier is too slow, $O(N^4)$. It seems however that MERA appears as a powerful tool. The main feature of MERA is the renormalization of the real space into the virtual space. This geometry reduces the complexity to the order of $O(N \log N)$. A basic method for simulating gaussian processes is by Cholesky decomposition which is of the order $O(N^3)$. Other more powerful methods scale as $O(N^2)$. In figure (4), we have plot the path and increments of a fractional motion. We calibrated the MERA by approximating the covariant matrix of the process. In figure (5), we have plotted a fit of the correlation $E(B_t^2)$. This correlation is evaluated by Monte Carlo and generating the process with MERA.

C. Matrix Product State Representation

Matrix Product States were originally understood as the ansatz for the density renormalization group algorithm [18]. The construction of these states has appeared, disappeared and reappeared many times through history under many different names such as finitely correlated states, Tensor trains, (complex) (quantum) hidden markov chains, in many different fields such as data science, quantum physics, statistical mechanics,... By consequence, we will focus on the form of interest for this paper. For a gaussian process $B_{T=t_N} = \sum_{j=1}^N X_{t_j}$, one may want to rewrite the joint probability distribution as follows,

$$\begin{aligned} P(X_{t_1} = x_1, \dots, P(X_{t_N} = x_N)) \\ = \int_{\mathbb{R}^N} d\vec{u} A^{(x_2)}(u_2, u_3) \dots A^{(x_{N-1})}(u_{N-1}, u_N) A^{(x_N)}(u_N) \end{aligned} \quad (5)$$

The Matrix Product State tensor $A^{(x_j)}(u_j, u_{j+1})$ is graphically represented in figure (5). If we decide to cut the circuit up to a height $N_\infty < \infty$, this circuit can be represented by

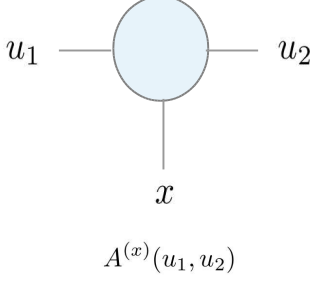


FIG. 7: Matrix Product States tensor

a Matrix Product State as shown in figure (2). It only rests us to precisely evaluate how the debt of the circuit N_∞ and the sample size are related to some error δ . The sample size should imply an error of at least the order $O(1/N)$. Furthermore, error due debt the circuit depends on the affine parameter $m_{\tau_0}^{\tau_k}$ which decays as $O((1/\tau)^{1-H})$. The quantification of the error is tricky. Ideally, we should convergence look at the convergence in the 1-norm $\|\cdot\|_1$ of the distributions. However, this is analytically not feasible. We could instead compare the covariance matrices. It seems the easiest to study the following error δ ,

$$\delta = \max_{0 \leq s \leq t} \left| E(B_s^{\text{fm}} B_t^{\text{fm}}) - \sum_{n_1, n_2=0}^{N_1, N_2} E(X_{t_{n_1}} X_{t_{n_2}}) \right| \quad (6)$$

with $t_{N_1}, t_{N_2} \rightarrow s, t$. The covariance of a gaussian process with N increments consists of at most N^2 . Clearly the circuit (2) is too "deep" as it contains at least N^4 parameters in some approximation. However, MERA suggest a reduction to $O(N \log N)$ parameters is possible.

III. MULTIFRACTAL PROPERTIES

Fractional Brownian motion is called unifractal. This property is coupled with the Hurst index H ,

$$B^{\text{fm}}(at) \stackrel{d}{=} a^H B^{\text{fm}}(t)$$

A more general property is multifractality. Rather than satisfying a global scaling with a unique affine parameter, there could be a distribution of many local scaling,

$$|X(t + a\Delta t) - X(t)| \stackrel{d}{=} M(a) |X(t + a\Delta t) - X(t)| \quad (7)$$

This time, however, $M(a)$ is a positive random variable which only depends on a and not t . This is achieved from our construction by the introduction a randomization in the Hurst index H inside the now-random variable $m_{\tau_k}^{\tau_{k+1}}$,

$$m_{\tau_k}^{\tau_{k+1}} = \gamma_k^{k+1} \exp(\log M(\tau_{k+1}) - \log M(\tau_k)) \quad (8)$$

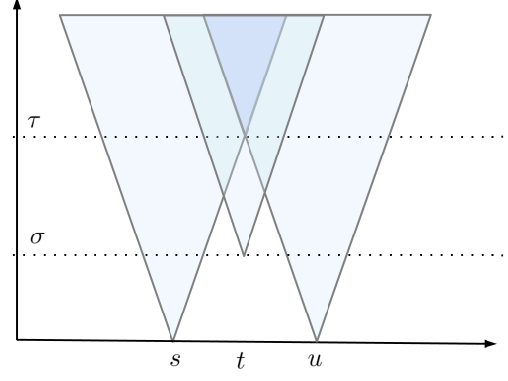


FIG. 8: The increments X_s and X_u depend solely on random variables inside lightcones starting respectively at s and u . Their correlation $g(\tau)$ is then determined by the overlap of the lightcones at (t, τ) . Per construction, the correlation satisfies a renormalization property as $g(\tau) = \left(\frac{\tau^2}{\sigma^2}\right)^{2H-2} g(\sigma)$.

The one-parameter random variable $M(\tau)$ satisfies the additional multiplicative property,

$$M(a\tau) \stackrel{d}{=} M_1(a)M_2(\tau) \quad (9)$$

with M_1 and M_2 independent random variables. More details and examples are given in the appendix (B). We can use the structure (1) to derive the following extension of the previous theorem to multifractal processes.

Theorem 2. *The joint probability distribution of the process $X(t) = \sum_{j=1}^N X_{t_j}$ constructed using $m_{\tau_k}^{\tau_{k+1}}$ as given by equation (8) and with random variables $M_j(\tau)$ satisfying property (9) implies the local scaling (7) and multifractality,*

$$X(at) \stackrel{d}{=} M(a)X(t)$$

Proof. Similarly to the proof for the fractional Brownian motion, it is sufficient to find scaling on the level of the correlations $E(X_{t_k} X_{t_l})$. The key intuition is illustrated in figure (8). Both $X_{s=t_k}$ and $X_{u=t_l}$ depend on random variable inside the light cones s and u respectively. Hence the correlations is determined by the random variable inside their intersection which is the light cone starting at $(t = |u - s|/2, \tau)$. In other words these random variable determine a new random variable $X_{t,\tau}$. Scaling is implied if,

$$X_{t,a\tau} \stackrel{d}{=} M(a)X_{t,\tau}$$

This is precisely implied by construction of $m_{\tau_k}^{\tau_{k+1}}$ and M in equations (8) and (9). Further technical details can be found in the appendix (B). \square

IV. CONCLUSION AND FURTHER DIRECTIONS

In this work, we constructed the joint probability measure of the increments of fractional brownian motion using the

framework of tensor network. This insight presented us on one hand with a new sampling method of fractional Brownian motion using the already known MERA, used in the study of quantum critical systems. Secondly, we show that such circuits present a novel pictorial representation of multifractal processes. In such representations, the connection between multifractality and renormalization emerges naturally. This network representation also presents a new insight of the multifractal processes. In the language of networks, the Hurst index is not unique anymore but the value on the different levels

of the circuit is sampled from a self-similar measure.

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- [1] Stephane Attal, Nadine Guillotin-Plantard, and Christophe Sabot. Central limit theorems for open quantum random walks and quantum measurement records. *arXiv:1206.1472*, 2012.
 - [2] Jan Beran. *Statistics for long-memory processes*, volume 61. CRC Press, 1994.
 - [3] Ton Dieker. Simulation of fractional brownian motion. *MSc theses, University of Twente, Amsterdam, The Netherlands*, 2004.
 - [4] CJ Evertsz and Benoit B Mandelbrot. Multifractal measures. *Chaos and Fractals, Springer-Verlag, New York*, 1992.
 - [5] Igor Vladimirovich Girsanov. On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability & Its Applications*, 5(3):285–301, 1960.
 - [6] Jutho Haegeman, Tobias J Osborne, Henri Verschelde, and Frank Verstraete. Entanglement renormalization for quantum fields in real space. *Physical review letters*, 110(10):100402, 2013.
 - [7] David Hochberg and Juan Pérez-Mercader. The renormalization group and fractional brownian motion. *Physics Letters A*, 296(6):272–279, 2002.
 - [8] Demetris Koutsoyiannis. The hurst phenomenon and fractional gaussian noise made easy. *Hydrological Sciences Journal*, 47(4):573–595, 2002.
 - [9] Benoit B Mandelbrot. Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. In *Multifractals and 1/ Noise*, pages 317–357. Springer, 1999.
 - [10] Benoit B Mandelbrot, Adlai J Fisher, and Laurent E Calvet. A multifractal model of asset returns. 1997.
 - [11] Benoit B Mandelbrot and John W Van Ness. Fractional brownian motions, fractional noises and applications. *SIAM review*, 10(4):422–437, 1968.
 - [12] Masahiro Nozaki, Shinsei Ryu, and Tadashi Takayanagi. Holographic geometry of entanglement renormalization in quantum field theories. *Journal of High Energy Physics*, 2012(10):1–40, 2012.
 - [13] David Perez-Garcia, Frank Verstraete, Michael M Wolf, and J Ignacio Cirac. Matrix product state representations. *arXiv preprint quant-ph/0608197*, 2006.
 - [14] Rudolf H Riedi. Multifractal processes. Technical report, DTIC Document, 1999.
 - [15] Ilya Sinayskiy and Francesco Petruccione. Open quantum walks: a short introduction. In *Journal of Physics: Conference Series*, volume 442, page 012003. IOP Publishing, 2013.
 - [16] Frank Verstraete, Valentin Murg, and J Ignacio Cirac. Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems. *Advances in Physics*, 57(2):143–224, 2008.
 - [17] Guifre Vidal. Entanglement renormalization. *Physical review letters*, 99(22):220405, 2007.
 - [18] Steven R White. Density matrix formulation for quantum renormalization groups. *Physical Review Letters*, 69(19):2863, 1992.
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Appendix A: Fractional Brownian Motion

Theorem. Given the join probability distribution

$$P(\vec{X}_{t_j, \tau_0}) = \sum_{j_1, \dots, j_{N_\infty}} P(\vec{X}_{t_{j_0}, \tau_0} | \vec{X}_{t_{j_1}, \tau_1}) P(\vec{X}_{t_{j_1}, \tau_1} | \vec{X}_{t_{j_2}, \tau_2}) \dots P(\vec{X}_{t_{j_{N_\infty-1}}, \tau_{N_\infty-1}} | \vec{X}_{t_{j_{N_\infty}}, \tau_{N_\infty}}) \quad (\text{A1})$$

which is constructed from the network whose structure is pictured in figure (2)

$$P(\vec{X}_{t_{j_k}, \tau_k} | \vec{X}_{t_{j_{k+1}}, \tau_{k+1}}) = \prod_l P(X_{t_l}, \tau_k | X_{t_l}, X_{t_{l+1}}, \tau_{k+1}) \quad (\text{A2})$$

with the transfer operations given by,

$$P(X_{t_l, \tau_k} | X_{t_l, \tau_{k+1}} = z_1, X_{t_{l+1}, \tau_{k+1}} = z_2) \propto \mathcal{N}(y; m_{\tau_k}^{\tau_{k+1}}(z_1 + z_2), \sigma_{\tau_k}^{t_k}) \quad (\text{A3})$$

The parameters are taken to be,

$$t_n = \epsilon n, \quad \tau_n = \sqrt{n\delta}, \quad \delta = \epsilon^2, \quad m_{\tau_j}^{\tau_k} = \left(\frac{\tau_k}{\tau_j}\right)^{H-1} \frac{\left[k \binom{2k}{k}\right]^{-1/2}}{\left[j \binom{2j}{j}\right]^{-1/2}}, \quad \sigma_{\tau_k}^{t_k} = \frac{1}{2} \sigma \sqrt{H|2H-1|\Gamma(1-H)^{-1}}$$

The variance $\sigma_{\tau_k}^{t_k}$ satisfies the boundary condition,

$$\sigma_{\tau_0}^{t_0} = \begin{cases} \sqrt{\epsilon} \sigma & H = 1/2 \\ \frac{1}{2} \sigma \sqrt{H|2H-1|\Gamma(1-H)^{-1}} & \text{else} \end{cases}$$

In the limit $N \rightarrow \infty$ and $t_N \rightarrow T$, the process $B_T^m = \sum_j X_{t_j}$ with joint probability distribution $P(X_{t_1}, \dots, X_{t_N})$, is then a fractional brownian motion with Hurst index H , which satisfies,

$$E(B_{t_{N_1}} B_{t_{N_2}}) = \frac{1}{2} \sigma^2 (t_{N_1}^{2H} + t_{N_2}^{2H} - |t_{N_1} - t_{N_2}|^{2H})$$

Proof. The calculations are more insightful if we keep figure (8) and the renormalization flow equation (4) in mind. The correlation is then determined by the overlap of the lightcones of X_{t_k} and X_{t_l} , which is another lightcone at $(|t_k - t_l|/2, \tau_n = \sqrt{|t_k - t_l|\epsilon/2})$. One can check that,

$$\begin{aligned} E(X_{t_k} X_{t_l}) &\propto \frac{1}{2} \sigma^2 \sum_{q=n}^{N_\infty} (m_{\tau_0}^{\tau_q})^2 \sum_{j=n}^q \binom{q}{j} \binom{q}{j-n} \\ &= \frac{1}{2} \sigma^2 \epsilon^2 \sum_{q=n}^{N_\infty} \tau_q^{2H-2} \left[\sum_{j=n}^q \binom{q}{j} \binom{q}{j-n} \right] \left[q \binom{2q}{q} \right]^{-1} \end{aligned} \quad (\text{A4})$$

Using Vandermonde Convolution's identity and Stirling's formula, we can approximate the binomial coefficients,

$$\sum_{j=n}^q \binom{q}{j} \binom{q}{j-n} = \binom{2q}{n}, \quad \binom{2q}{q-n} \binom{2q}{q}^{-1} \approx e^{-n^2/q}$$

Introducing a rescaling $\gamma(q) = q/n^2$, and using the identity followed by approximation above simplifies the equation (A4),

$$\sum_{q=n}^{n^2} \tau_n^{2H-2} \binom{2q}{q-n} \binom{2q}{q}^{-1} = \left(\sum_{q=n}^{N_\infty} n^{-2} \exp(-\gamma(q)^{-1}) \gamma(q)^{H-2} \right) t_n^{2H-2}$$

In the limits $n^2/N_\infty \rightarrow 0$ and $n \rightarrow \infty$, the expression between brackets converges to the Gamma function,

$$\sum_{q=n}^{N_\infty} n^{-2} \exp(-\gamma(q)^{-1}) \gamma(q)^{H-2} \approx \int_{n^2/N_\infty}^n du e^{-u} u^{-H} \rightarrow \Gamma(1-H)$$

Combining the results yields the correlation,

$$E(X_{t_k} X_{t_l}) \propto \frac{1}{2} \sigma^2 \epsilon^2 t_{|k-l|}^{2H-2}$$

For large N and small ϵ , we can approximate the double sum by a double integral,

$$\sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \epsilon^2 \tau_{|k-l|}^{2H-2} \approx \int_0^{t_{N_1}} du_1 \int_0^{t_{N_2}} du_2 |u_1 - u_2|^{2H-2} = \frac{1}{2} \sigma^2 (t_{N_1}^{2H} + t_{N_2}^{2H} - |t_{N_1} - t_{N_2}|^{2H})$$

from which the claim follows. \square

Appendix B: Multifractal process

A key component for the introduction of multifractal measures, are the so-called self-similar measures. A detailed introduction can be found in [10, 14]. Define the set \mathcal{S} to consist of all similitude transformation, i.e. translation and homothetic transformations.

Definition 3. Given $\mu : [0, T] \rightarrow [0, 1]$ a random measure, which satisfies,

1. For all similitudes $S \in \mathcal{S}$, for any interval $I_1 \subset I_2$, the ratios,

$$\frac{\mu(SI_1)}{\mu(I_1)} \stackrel{d}{=} \frac{\mu(I_1)}{\mu(I_1)} \quad (\text{B1})$$

are equal in distribution as long as $I_1, I_2, SI_1, SI_2 \subset [0, T]$.

2. For all decreasing sequences of compact intervals $I_1 \subset I_2 \subset \dots \subset I_n \subset [0, T]$, the ratios,

$$\frac{\mu(I_1)}{\mu(I_2)}, \dots, \frac{\mu(I_{n-1})}{\mu(I_n)} \quad (\text{B2})$$

are statistically independent.

then, the measure μ is called self-similar.

The first property (B1) implies the existence of a random variable M such that,

$$\mu[0, ct] \stackrel{d}{=} M(c) \mu[0, t], \quad 0 \leq ct, t \leq T \quad (\text{B3})$$

From the second property (B1), we also derive that the random variable must satisfy a multiplicative property. Taking $0 \leq c_1, c_2 \leq 1$ and $0 \leq t \leq T$,

$$\frac{\mu[0, c_1 c_2 t]}{\mu[0, t]} = \frac{\mu[0, c_1 c_2 t]}{\mu[0, c_2 t]} \frac{\mu[0, c_2 t]}{\mu[0, t]}$$

Hence, by the corollary of the first property of self-similar measures (B3),

$$M(c_1 c_2) \stackrel{d}{=} M'(c_1) M''(c_2)$$

The second property (B1) implies that M' and M'' are independent. The existence of such random variable plays a central role when introducing multifractality. Before jumping to the derivation of our result, we illustrate this property with two examples.

Example 4. For $t > 1$, define $M(t)$,

$$M(t) = \exp \left(\sigma B_{\log(t)} - \frac{\sigma^2}{2} \log(t) \right)$$

with the brownian motion $B_{\log(t)} = \int_0^{\log(t)} dB_s$. From the indepenence of the increments of Brownian motion, we readily derive,

$$M(ct) = \exp \left(\sigma B_{\log(c)} - \frac{\sigma^2}{2} \log(c) \right) \exp \left(\sigma B_{\log(t)} - \frac{\sigma^2}{2} \log(t) \right) = M(c) M(t)$$

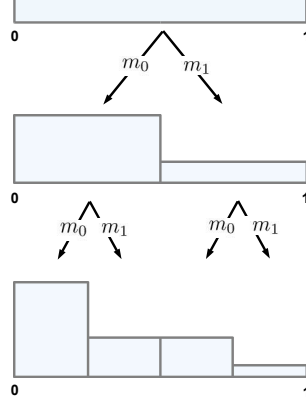


FIG. 9: Illustration of the multiplicative cascade for constructing the binomial multifractal measure in example (5). In the second step, the masses of each cell are respectively m_0^2, m_0m_1, m_0m_1 and m_1^2 .

Example 5 (Binomial Measure). *As we will see this example does not satisfy the properties for all c, t , but is, however, very insightful. The binomial measure is introduced as the limit of an elementary iterative procedure called a multiplicative cascade. As illustrated in figure (9), the idea is to iteratively divide the interval, for sake of simplicity $[0, 1]$, into b -adic cells of length $1/b$. At each step, the mass is multiplied by a factor depending on the location of the cell. For example after k steps, the mass of the cell $t = \sum_{j=1}^k \eta_j 2^{-j}$ with length $\Delta t = 2^{-k}$ is,*

$$\mu[t, t + \delta t] = M(\eta_1)M(\eta_1, \eta_2) \dots M(\eta_1, \dots, \eta_k)$$

Additionally, we can choose each $M(\eta_1, \dots, \eta_k)$ to be independent.

The most simple case is to choose a unique weight m_0 . At each step of the interaction one multiplies by m_0 if we take the left cell and $m_1 = 1 - m_0$ for the right cell. By induction for $t_{n+1} = t_n + \eta_{n+1} 2^{-(n+1)}$ and $\Delta_n = 2^n$,

$$\mu[t_{n+1}, t_{n+1} + \Delta t_{n+1}] = \mu[t_n, t_n + \Delta t_n] \begin{cases} m_0 & \text{if } \eta = 0 \\ m_1 & \text{if } \eta = 1 \end{cases}$$

Let us repeat the procedure $N \gg 1$ times, and consider two dyadic numbers c and t . One should see that by the self-similarity of the construction, the multiplicative property follows,

$$\mu[ct, ct + \Delta t_N] = \mu[c, c + \Delta t_N] \mu[t, t + \Delta t_N]$$

Hence, in this example we can construct,

$$M(t) = \lim_{N \rightarrow \infty} \mu[t, t + \Delta t_N]$$

More information about this construction can be found [4].

Theorem 6. *Given the join probability distribution with the same structure as in equation (A1) and transfer operations (A2) with parameters,*

$$t_n = \epsilon n, \tau_n = \sqrt{n\delta}, \delta = \epsilon^2, m_{\tau_0}^{\tau_n} = \exp \left(\frac{1}{2} \log \left[\frac{\tau_n^{-4} M(\tau_n)}{\epsilon} \right] \right) \left(\frac{2n}{n} \right)^{-1/2}$$

$$m_{\tau_k}^{\tau_l} = \exp \left(\frac{1}{2} \log \left[\frac{\tau_l^{-4} M(\tau_l)}{\tau_k^{-4} M(\tau_k)} \right] \right) \frac{\left[k \binom{2k}{k} \right]^{-1/2}}{\left[j \binom{2j}{j} \right]^{-1/2}}, \quad \sigma_{\tau_k}^{t_k} = \sigma$$

The continuous one-parameter random variable $M(\tau_n)$ satisfies the multiplicative property,

$$M(c\tau_n) \stackrel{d}{=} M'(c)M''(\tau_n) \tag{B4}$$

where $M'(c)$ and $M''(\tau_n)$ are independent. The process $Y_T = \sum_j X_{t_j}$ with joint probability distribution $P(X_{t_1}, \dots, X_{t_N})$, is then a multifractal process which satisfies the scaling property,

$$E(Y^m(cT)) = E(M^m(c)) E(Y^m(T))$$

Proof. We show for all $m \geq 1$,

$$E(Y_{ct}^m) = E(M(c)^m) E(Y_t^m)$$

Let us first fix the value of the random variable $M(\cdot)$ evaluated at different times. The process is then gaussian for all such values. Hence, the moments are zero for odd m and powers of the variance for m even. Similary to the case of fractional Brownian motion, we evaluate the second moment. One can check that,

$$\tilde{E}(X_{t_k} X_{t_l}) = \sigma^2 \epsilon^2 \sum_{q=n}^{N_\infty} \tau_q^{-2} M(\tau_q) \left[\sum_{j=n}^q \binom{q}{j} \binom{q}{j-n} \right] \left[q \binom{2q}{q} \right]^{-1} \quad (\text{B5})$$

The expectation $\tilde{E}(\cdot)$ was taken with respect to the gaussian random variables ξ_{t_k, τ_l} , excluding $M(\cdot)$. Repeating the procedure of Stirling's approximation and change of variable, we simplify the density,

$$\begin{aligned} \sum_{q=n}^{N_\infty} \tau_q^{-2} M(\tau_q) \left[\sum_{j=n}^q \binom{q}{j} \binom{q}{j-n} \right] \left[q \binom{2q}{q} \right]^{-1} &\approx \sum_{q=n}^{N_\infty} n^{-2} \exp(-\gamma(q)^{-1}) \gamma(q)^{-2} M(\tau_{\gamma n^2}) t_n^{-2} \\ &\approx \int_{1/n}^{N_\infty/n^2} d\gamma \exp(-\gamma(q)^{-1}) \gamma(q)^{-2} M(\gamma t_n) t_n^{-2} \end{aligned}$$

where we used $t_n \approx n\epsilon$. Hence,

$$\tilde{E}(Y_T^2) = \kappa \int_0^T du_1 \int_0^T du_2 \int_0^\infty d\gamma \exp(-\gamma^{-1}) \gamma^{-2} M(\gamma|u_1 - u_2|) |u_1 - u_2|^{-2}$$

for some constant κ . As higher even moments are propotional to powers of the second moment, we readily see after taking the expectation w.r.t. the distribution of $M(\cdot)$ and using the multiplicative property (B4),

$$E(Y_T^{2m}) \propto \kappa^m \int_0^T d\vec{u} \int_0^\infty d\vec{\gamma} \exp\left(-\sum_j \gamma_j^{-1}\right) \left(\prod_j \gamma_j |u_{2j-1} - u_{2j}|\right)^{-2} E\left(\prod_j M(\gamma_j)\right) E\left(\prod_j M(|u_{2j-1} - u_{2j}|)\right)$$

Using the multlicative property, this expression yields the sought property,

$$E(Y_{cT}^m) = E(M_c^m) E(Y_T^m)$$

from which the claim follows. □

$$P(Y_{\tau_k}^{t_k}, Y_{\tau_k}^{t_{k+1}} | Y_{\tau_{k+1}}^{t_k} = z_1, Y_{\tau_{k+1}}^{t_{k+1}} = z_2) \propto \mathcal{N}(y; m_{\tau_k}^{\tau_{k+1}}(z_1 + z_2), \sigma_{\tau_k}^{t_k}, \sigma_{\tau_k}^{t_{k+1}})$$
